1027 -
Investigating the path of light in Fata Morgana

By Michael Yang

About this project

International Baccalaureate Extended Essay on mathematics

The overall goal of the report is to investigate and model the phenomenon of **Fata Morgana**, also known as superior mirage.

The predominant means of which is by utilizing **Fermat's principle of least time** and **Snell's law**, which is derived from it.

Natural Phenomenon

Air near the surface is significantly colder than that of the atmosphere. This happens near large sheets of ice or near the surface of oceans.

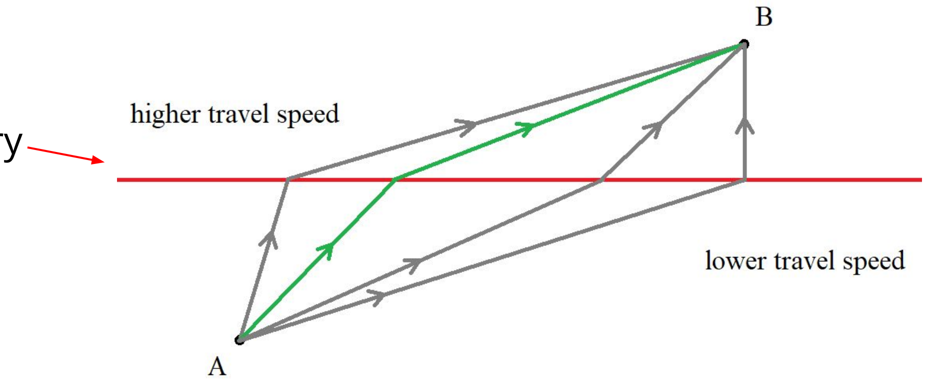
Leads to **refraction**, or bending of light.

Timpaananen. (2012). *Fata Morgana Example* [Photograph]. Wikimedia Commons. Retreived September 28, 2020, from https://commons.wikimedia.org.

Refraction

Optical phenomenon that occurs whenever there is a difference in the **travel speed** of light between two media.

boundary



Green: optimal path (shortest travel time)

This can be intuitively explained with Fermat's principle: the path that a ray takes to travel between two points will always be one with the smallest travel time.

Snell's law

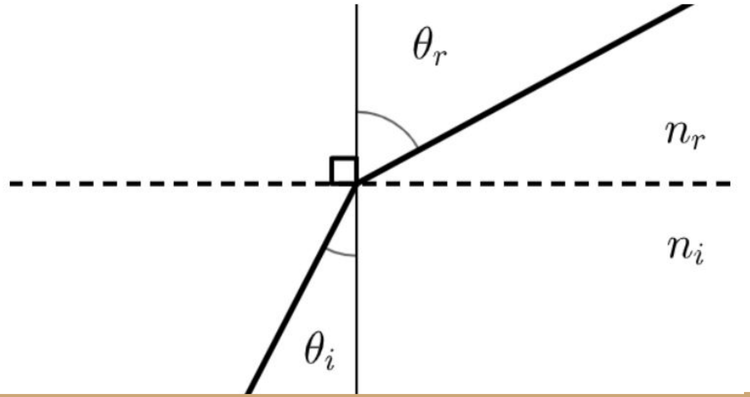
The relative travel speed of light is given its own measure, the **refractive index**:

$$n = \frac{c}{v}$$

c: speed of light in a vacuum

v: speed of light in the medium

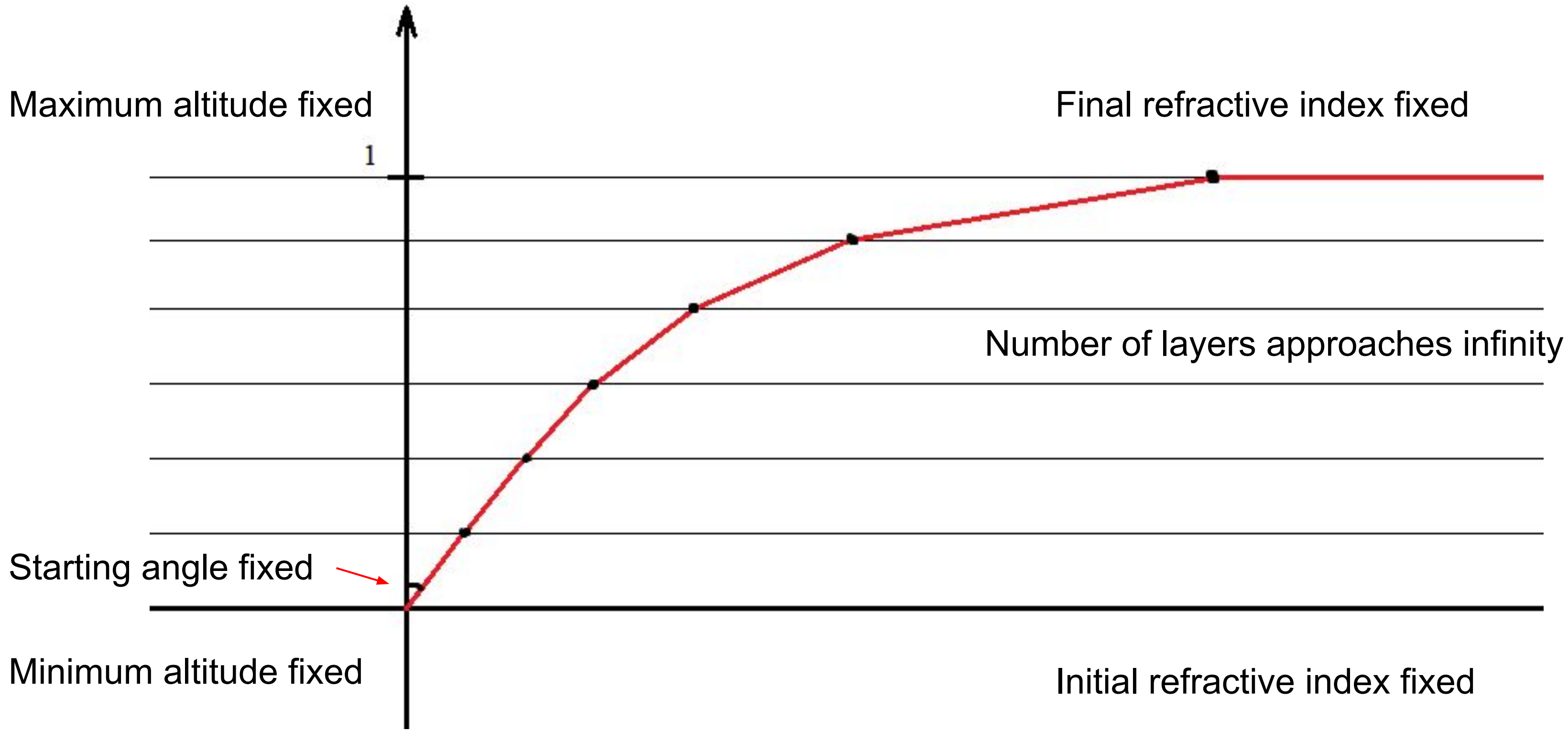
The way that light bends follows **Snell's Law**: $n_i \times \sin\theta_i = n_r \times \sin\theta_r$



The **aim** of the project is to investigate the way that the sequence of refractive indices (arithmetic/geometric for example) within a constricted space will affect the resulting path of light.

This is done by first constructing a relevant model for the phenomenon (next slide).

Constructing the model



Method

The coordinates for a set of points which represent the intersections between the beam of light and the layer boundaries are expressed as follows:

Initial refractive index → n_0; Starting angle → θ_0

$$x_a = \frac{1}{k}\sum_{i=1}^{a} \tan\left[\arcsin\left(\frac{n_0 \times \sin\theta_0}{n_i}\right)\right]$$

Number of layers → k

$$y_a = \frac{a}{k}$$

(For the a^{th} layer above the x-axis)

The derivation of the x-coordinate can be shortly summarized in the next slide.

Method (cont.)

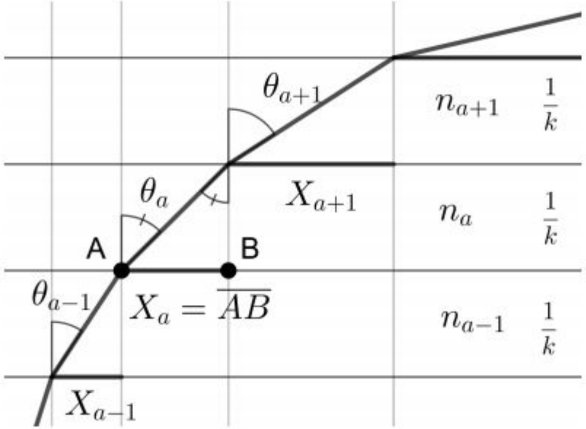
$$x_a = X_1 + X_2 + X_3 + \cdots + X_{a-1} + X_a$$

$$= \sum_{i=1}^{a} X_i$$

Horizontal lengths between subsequent points of intersection

$$X_a = \frac{\tan \theta_a}{k}$$

$$x_a = \frac{1}{k} \sum_{i=1}^{a} \tan \left[\arcsin \left(\frac{n_0 \times \sin \theta_0}{n_i} \right) \right]$$



Solved using Snell's law from the equation

$$n_a \times \sin \theta_a = n_0 \times \sin \theta_0$$

The remaining variable of the equation is the refractive index of each layer (present in the expressions as n_i), which will be expressed by the sequences that are formulated next.

Boundaries of sequences

To ensure that the beam of light is parallel to the horizontal by the maximum altitude, the following boundaries are determined:

- The refractive index of the initial layer $$n_0 = \frac{1}{\sin \theta_0}$$
- The refractive index of the final layer $$n_k = 1$$

Defining sequences

The arithmetic sequence is first to be expressed:

$$n_a = \frac{1}{\sin\theta_0} + \frac{a}{k}\left(1 - \frac{1}{\sin\theta_0}\right)$$

The geometric sequence is then expressed:

$$n_a = \sin^{\frac{a}{k}-1}\theta_0$$

Coordinates expressed explicitly

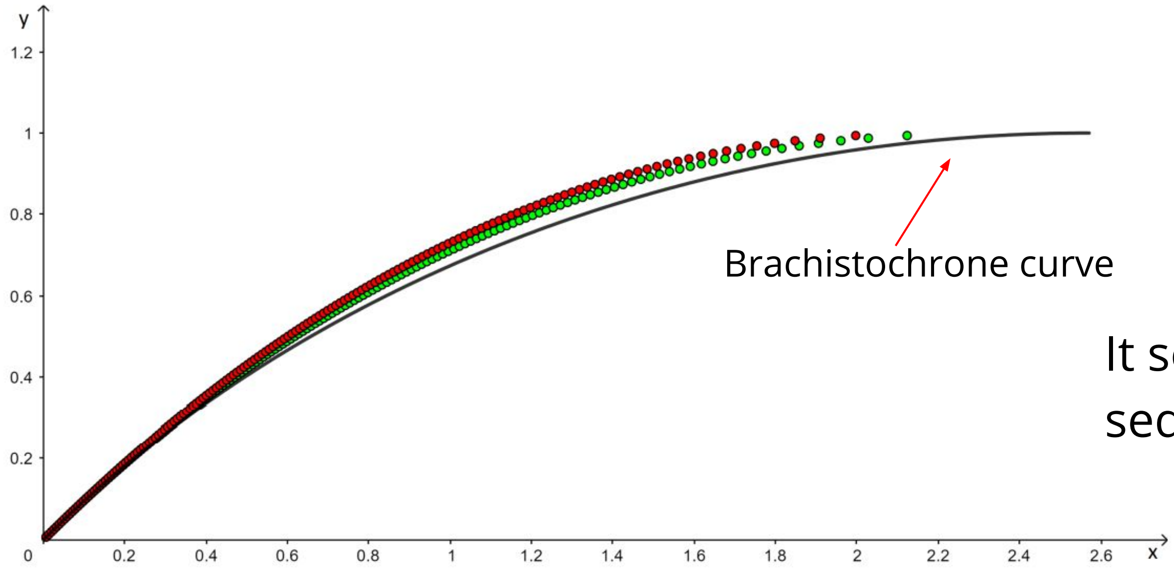
Arithmetic: $$x_a = \frac{1}{k}\sum_{i=1}^{a} \tan\left[\arcsin\left(\frac{\sin\theta_0}{1+\frac{i}{k}(\sin\theta_0 - 1)}\right)\right]$$

Geometric: $$x_a = \frac{1}{k}\sum_{i=1}^{a} \tan\left[\arcsin\left(\sin^{1-\frac{i}{k}}\theta_0\right)\right]$$

$$y_a = \frac{a}{k}$$ (same for both of the sequences)

Comparison #1

The resulting paths closely resembles that of a **brachistochrone**, which would lead to our next goal: finding the sequence of refractive indices that would result in this curve.



Red: arithmetic sequence

Green: geometric sequence

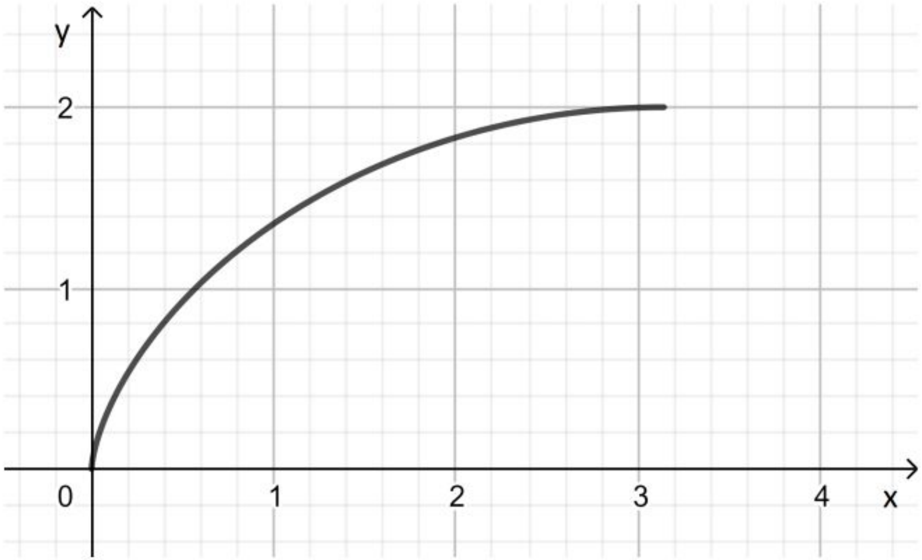
It seems that the geometric sequence is closer to the curve

Brachistochrone curve

Relating to Fermat's principle of least time is the Brachistochrone problem:

"What is the path of least travel time of a ball that rolls from point A to B which is below it but not directly?"

The solution is the **brachistochrone** curve.



(The curve in this case is the vertical inverse of the path that a ball would roll along in order to compare it to the previous slide.)

Modelling after the brachistochrone

Using the original prompt of the brachistochrone problem, we notice that the **conservation of mechanical energy** takes place when the ball rolls down.

Thus:

$$E_P = E_K \rightarrow mgh = \frac{1}{2}mv^2 \rightarrow y = \frac{1}{2g}v^2 \rightarrow v \propto \sqrt{y}$$

From the proportionality, we can construct a function and then discretize it in order to formulate a sequence of velocities within the media at different altitudes (y-values).

Solving for the sequence of velocities

Boundary conditions solved for velocity according to the conditions for the indices:

$$n = \frac{c}{v} \quad \rightarrow \quad v_0 = c \sin\theta_0 \quad \text{(when } y = 0\text{)}$$

$$v_k = c \quad \text{(when } y = 1\text{)}$$

Function of velocity that somewhat satisfies the proportionality and the boundary conditions:

$$v = c\sqrt{(1 - \sin^2\theta_0)y + \sin^2\theta_0}$$

Coordinates for the sequence of velocity

Expressing the coordinates of the points for the sequence of velocity:

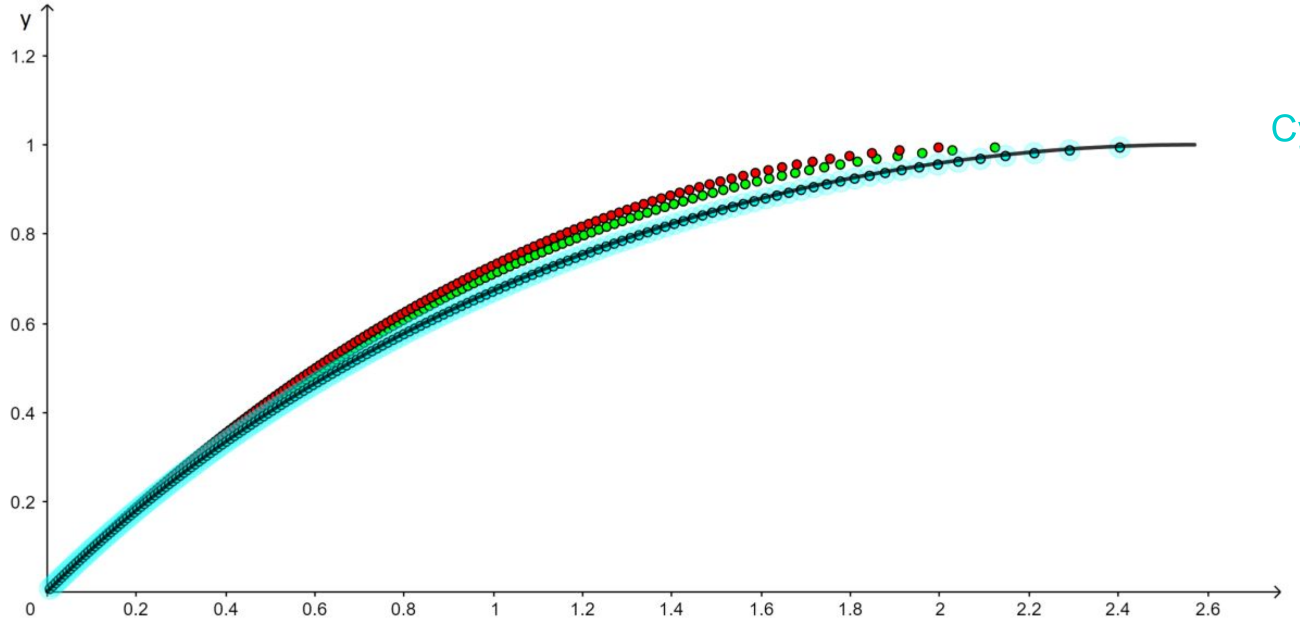
$$x_a = \frac{1}{k}\sum_{i=1}^{a} \tan\left[\arcsin\left(\frac{n_0 \times \sin\theta_0}{n_i}\right)\right]$$

$$= \frac{1}{k}\sum_{i=1}^{a} \tan\left[\arcsin\left(\frac{v_i \times \sin\theta_0}{v_0}\right)\right]$$

$$x_a = \frac{1}{k}\sum_{i=1}^{a} \tan\left[\arcsin\left(\sqrt{(1-\sin^2\theta_0)\frac{i}{k} + \sin^2\theta_0}\right)\right]$$

Comparison #2

When all of the sets of points are plotted to the same graph:



Cyan: sequence of velocity

Evaluation

1. Qualitative comparison by graph (previous slide)
2. Quantitative comparison by inspecting the average of the differences between the points and the brachistochrone curve:

$$\bar{\gamma} = \left[\left| C\left(\frac{1}{k}\right) - x_1 \right| + \left| C\left(\frac{2}{k}\right) - x_2 \right| + \ldots + \left| C\left(\frac{a-1}{k}\right) - x_{k-1} \right| \right] \times \frac{1}{k-1}$$

$$= \frac{1}{k} \sum_{a=1}^{k-1} \left| C\left(\frac{a}{k}\right) - x_a \right|$$

(where *C(y)* is the cartesian expression for the brachistochrone curve)

Evaluation (cont.)

The values are calculated for all sequences (with set amounts of layers *k*)

k	Arithmetic n sequence	Geometric n sequence	v sequence
10	0.0403	0.0137	0.0515
20	0.0636	0.0359	0.0296
30	0.0727	0.0448	0.0209
40	0.0775	0.0496	0.0162
50	0.0805	0.0526	0.0132
60	0.0826	0.0546	0.0112
70	0.0841	0.0562	0.0097
80	0.0853	0.0573	0.0086
90	0.0862	0.0582	0.0077
100	0.0869	0.0589	0.0070

Value for the sequence of velocity approaches zero

→ indicates the similarities between the curve and the points

Conclusion

- The set of points with an arithmetic sequence of refractive indices has the largest difference from the brachistochrone curve,
- then comes the set of points with a geometric sequence of refractive indices,
- and finally the set of points obtained via inspection of the original brachistochrone problem (with the proportionality between the altitude and the travel speed of light) very closely resembles the original curve.

Although we cannot be sure that such a proportionality works in real life, it is evident that regardless of the sequence of refractive indices, the path of light will somewhat resemble the curve nonetheless.